Matrix Representation of Special Relativity

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Abstract

I compare the matrix representation of the basic statements of Special Relativity with the conventional vector space representation.

It is shown, that the matrix form reproduces all equations in a very concise and elegant form, namely: MAXWELL equations, LORENTZ-force, energy-momentum tensor, DIRAC-equation and LAGRANGIANS.

The main thesis is, however, that both forms are nevertheless not equivalent, but matrix representation is superior and gives a deeper insight into physical reality, because it is based on much less assumptions. It allows a better understanding of MINKOWSKI spacetime on the basis of matrix algebra.

An escpecially remarkable result of the consequent usage of this algebraic concept is the formulation of DIRACs equation in a novel matrix form. This equation explains the non-existence of righthanded neutrinos and can be generalized to include a new variant of YANG-MILLS gauge fields, which possibly express unified electro-weak interactions in a new way.

Contents

1 Introduction

The possibility to represent MINKOWSKI spacetime vectors with 2x2-matrices is well-known since the 1920ies (e.g. [10], [8], pp. 61). It is a consequence of the fact, that the LORENTZ-group is homomorphic to the group of unimodular binary matrices $SL(2, \mathcal{C})$.

This matrix representation is mostly used to show, how covariant equations for spinors can be derived. But it is widely ignored, that on the other hand, this also can lead to another description of spacetime

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itself. There also seems to exist a general consensus, that both representations (matrix form and usual component form) are actually equivalent methods to express the equations of Special Relativity, and consequently the matrix form is used very rarely in publications.[1]

One principal reason for this is surely the fact, that conventional component formulas can be formally applied to an arbitrary number of dimensions of the vector space, while the matrix form is only possible for the *four-dimensional* case.

In this article I show, that the presumed equivalence of both forms is not true. Although the equations are isomorphic (otherwise they would be wrong), significantly less prerequisites are needed to derive them for the matrix form.

The most important prerequsite is the existence of a *metric tensor* with the signature $(+---)$, that has to be postulated for the vector space, but it is automatically determined for the matrix formalism. In principle, any metric signature would be conceivable for the vector space. Since the metric tensor is - at least implicitely - contained in every relativistic equation, this statement shows, that the matrix form is superior.

Additionally, the homogeneous MAXWELL equations, which have to be introduced independently in SRT in component form, are a direct consequence of the inhomogeneous MAXWELL eqs. here.

The last and most important argument gives the reformulation of the DIRAC equation in matrix form. All arbitrary free parameters without physical content, which arise in the 4-spinor form, vanish here, because the remaining similarity transformations can be understood as *gauge transformations*.

Thus I propose a *change of perspective* here: Theoretical physicists should consider, that the physical spacetime primarily is a *matrix algebra* and the component formulation is only a derived one, which has several disadvantages. I will denote this perspective as "matrix spacetime" (MST) compared to "vector spacetime".[2]

Please note also that, if this point of view is adopted, this is *not only a formal aspect*, but it has far-reaching consequences for many other physical theories. E.g. General Relativity must be modified from its original form, since the metric tensor cannot be used as field variable.[3] Also all theories with more than four spacetime dimensions (string theory, etc.) are obviously excluded.

This quite *restrictive* point of view must be seen as an advantage. One may compare this with the principle of *special relativity.* It is so powerful, *because* it is so restrictive, namely it rules out all not covariant eqs. The same holds for the *gauge invariance* principle.

As long as there is *no real evidence* for any extra dimensions, this description of the physical world has at least a to be considered as *possible* and worth discussing.

This new perspective also may lead to new theories, e.g. if possible generalizations of this form are considered. Also, one might look for an underlying *spinor structure* for the matrix algebra, which is e.g. the main thesis of the "twistor-theory" presented in [9] (Vol. II) but has not led to a satisfactory physical theory yet.

In conclusion I have to say, that many of the equations presented here, can also be found scattered in other publications. However, their derivation and notation here is surely often more concise and straight forward.

New in any case, is the notation of DIRACs eq. as "matrix equation". Also the corresponding LAGRANGIAN, I have not found in another publication. This new form perhaps allows new insights in particle physics, esp. *unified ectro-weak theory.*

[1] One of the first fundamental papers on this topic is [11], where the idea of *em. gauge symmetry* was invented and some of the concepts and eqs. below can be found, but in a quite unusual notation for contemporary readers. Also, he focuses there on gravitation and curved space-time and many of his presumptions were invalidated if the following years.
A newer, quite voluminuous work is [9], where the authors try to give a fundamental overview from a mathematical point of view. They also focus on GR and various kinds of generalizations.
In neither of both works the rigorous physical interpretation is adopted, which is proposed here.

[2] Compare e.g. [4], where a similar concept with a four-dimensional algebra based on the CLIFFORD-matrices is presented. He uses a similar term "spacetime algebra" (STA).

[3] The 4 basis matrices $\{\tau_\mu\}$ introduced below, or equivalently the 16 coefficients a_μ^ν, which play the role of *tetrades*, have to be used instead. More detailled discussions of this can be found again in [11] and [9]

2 Matrix Representation of Minkowski-Vectors

Let me start with the 4-dimensional vector space of real numbers $V^4 = \{(x^0, x^1, x^2, x^3)\}$. This can be mapped one-to-one to the set of hermitean 2x2 matrices $\mathbf{M} = \{\mathbf{x}\}$, when a basis of 4 linearly independent hermitean matrices $\tau_\mu = (\tau_0, \ldots \tau_3)$ is given by (as usual, over double upper and lower indices $\mu = 0, \ldots 3$ is to sum):

$$\mathbf{x} = x^\mu \tau_\mu. \tag{1}$$

These hermitean matrices $\mathbf{x}$ build a well defined subset of the binary matrix algebra. In the following they are denoted as Minkowski-*matrices* and represented by **boldface** letters (except the Greek letters ρ, τ, σ and the partial operator ∂).

Since this is a *one-to-one map*, it is clear that all relations written in one form can also be transcribed into the other, and in principle no form can be given preference.
However, the crucial difference is, that one has to put a postulated *metric tensor* on top of the vector space, to define a *vector norm* and get covariant equations there.
As shown below, for the matrix representation, the existence and form of this tensor is a natural consequence of the algebraic structure.

For binary matrices holds:[4] $\bar{\mathbf{x}} = x^\mu \bar{\tau}_\mu$ and consequently the *matrix determinant* naturally defines a bilinear norm in (x^μ). This norm can now be identfied with the norm of the vector space. This is only for 2x2 matrices possible, and vector dimensions greater than four are excluded.

The metric tensor $g = (g_{\mu\nu})$ is then given by:

$$|\mathbf{x}| = x^\mu x^\nu \underbrace{\frac{1}{2}\mathcal{T}(\tau_\mu \bar{\tau}_\nu)}_{\stackrel{def}{=} g_{\mu\nu}} = x^\mu x^\nu g_{\mu\nu}. \tag{2}$$

Obviously, symmetry follows $g_{\mu\nu} = g_{\nu\mu}$ and all are real numbers, as required.

On the other hand, the four matrices τ_μ (like every hermitean matrix) can be expressed as linear combinations of the 3 Pauli-matrices $\sigma_1, \sigma_2, \sigma_3$ and a fourth matrix $\sigma_0 \stackrel{def}{=} I = \binom{10}{01}$:

$$\tau_\mu = a_\mu^\nu \sigma_\nu, \tag{3}$$

with 16 real coefficients a_μ^ν (for the vector space components this is to regard as a coordinate transformation: $x^\mu \to a_\nu^\mu x^\nu$).
Then follows from the known characteristics of the Pauli-matrices:[5]

$$g_{\mu\nu} = a_\mu^0 a_\nu^0 - a_\mu^1 a_\nu^1 - a_\mu^2 a_\nu^2 - a_\mu^3 a_\nu^3 = a_\mu^\lambda a_\nu^\delta g_{\lambda\delta}^{(0)}, \tag{4}$$

with $g^{(0)} = (g_{\lambda\delta}^{(0)}) = diag[+1, -1, -1, -1]$ as conventional Minkowski *metric tensor*. From this equation follows, that all possible metric tensors are transformations of $g^{(0)}$ and locally this metric can always be chosen. If the restriction of metric invariance ($g = g^{(0)}$) is made, then the (a_μ^ν) are identical to the Lorentz-*group*.

Consequently for simplification, the set of Pauli-matrices σ_μ is used in the following as basis. In this case the components can be simply recovered from the matrix form $\mathbf{x} = x^\mu \sigma_\mu$ by

$$x^\mu = \frac{1}{2}\mathcal{T}(\mathbf{x}\sigma_\mu) \qquad \Longleftrightarrow \qquad \mathbf{x} = x^\mu \sigma_\mu. \tag{5}$$

Explicitely it has the simple form $\mathbf{x} = \begin{pmatrix} t+z, & x-iy \\ x+iy, & t-z \end{pmatrix}$.

Because the matrix algebra includes addition and subtraction operations, also trivially the symmetry under *spacetime translations* holds, i.e. it shows the complete Poincare group symmetry.

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[4] the "bar" operation stands for matrix adjungation and $|\tau|$ for the determinant of the matrix τ, i.e. $|\tau|\tau^{-1} = \bar{\tau}$ holds. $\mathcal{T}(\tau)$ here denotes the scalar trace of τ, and from $\tau\bar{\tau} = \bar{\tau}\tau = |\tau| I$ follows $|\tau| = \frac{1}{2}\mathcal{T}(\tau\bar{\tau})$.

[5] with the usual representation $\sigma_1 = \binom{0,1}{1,0}, \sigma_2 = \binom{0,-i}{i,\ \ 0}, \sigma_3 = \binom{1,\ \ 0}{0,-1}$ one easily checks for all pairs $\mu, \nu = 0, \ldots, 3$ the orthogonality relation: $\sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu = I g_{\mu\nu}^{(0)}$.

3 Transformations and Covariant Forms

A LORENTZ-*transformation* is represented here by an unimodular 2×2 matrix $T \in SL(2, \mathcal{C})$, $|T| = 1$ and a MINKOWSKI-matrix transforms with:[6]

$$\mathbf{x} \to \mathbf{x}' = T\mathbf{x}T^\dagger, \tag{6}$$

which obviously preserves the hermitecity and the MINKOWSKI-invariant $|\mathbf{x}|$. It has of course 6 free real (3 complex) parameters.[7]

The general product $\mathbf{A\bar{B}}$ (obviously $\mathbf{AB}$ is not covariant under proper LT) of any two MINKOWSKI-matrices $\mathbf{A}, \mathbf{B}$ is then apparently a covariant matrix, because it transforms with:

$$\mathbf{A\bar{B}} \to (T\mathbf{A}T^\dagger)(\bar{T}^\dagger\mathbf{\bar{B}}\bar{T}) = T(\mathbf{A\bar{B}})\bar{T}. \tag{7}$$

The general scalar-product is the invariant expression, which is evidently always real:[8]

$$\frac{1}{2}\mathcal{T}(\mathbf{A\bar{B}}) = A_\mu B^\mu. \tag{8}$$

Space rotations, as important special case, are the subgroup of matrices, obeying $T^\dagger = \bar{T}(\equiv T^{-1})$. They additionally preserve the trace, which represents the time component $x^0 = \frac{1}{2}\mathcal{T}(\mathbf{x})$.[9]

Another important tranfsormation, which cannot be represented with any matrix T of this group, is *spatial inversion* $\mathcal{P}$. It is obviously described by[10]

$$\mathbf{x} \to \mathbf{x}_{sp} = \mathbf{\bar{x}}. \tag{9}$$

It is remarkable, that $\mathcal{P}$ is closely connected to the matrix multiplication order, since a general covariant equation of the form $\mathbf{A\bar{B}} = C$ transforms to $\mathbf{\bar{B}A} = \bar{C}_{sp}$.

4 Relativistic Electromagnetics

4.1 MAXWELL-Equations

In conventional **component notation** one starts with the antisymmetric field tensor $F_{\mu\nu} = -F_{\nu\mu}$, which is composed from electric and magnetic field vectors $\vec{E}, \vec{B}$:

$$F_{01} = E_1, \ldots \quad \text{and} \quad F_{12} = B_3, \ldots, .$$

Then the 4 *inhomogeneous* MAXWELL-eqs. are (with J_μ as 4-vector of current, see e.g. [1], p. 42)

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = J_\mu.$$

Additionally, the 4 *homogeneous* eqs. must be postulated:[11]

$$\frac{\partial F_{\mu\nu}}{\partial x_\sigma} + \frac{\partial F_{\nu\sigma}}{\partial x_\mu} + \frac{\partial F_{\sigma\nu}}{\partial x_\nu} = 0.$$

In the **matrix form** there is only <u>*one matrix equation*</u>, which includes either homog. and inhomog. eqs.:[12]

$$\underline{\partial F = \mathbf{J}}. \tag{10}$$

[6] $T^\dagger$ denoting the conjugate transpose (or hermite conjugate) of T.

[7] It is easy to show, that this group is *homomorphic* to the restricted LORENTZ-group and the homomorphism possesses the *kernel* $T \in \{I, -I\}$ (see e.g. [9], pp. 16).

[8] Above product matrix (7), can be decomposed into two covariant expressions: a scalar commutator (which is this scalar-product) and a traceless anti-commutator $\mathbf{A\bar{B}} = \frac{1}{2}(\mathbf{A\bar{B}} + \mathbf{B\bar{A}}) + \frac{1}{2}(\mathbf{A\bar{B}} - \mathbf{B\bar{A}})$.

[9] Since T is then is a similarity transformation, $T\mathbf{x}T^{-1}$, it is also clear that both eigenvalues of $\mathbf{x}$ are invariant.

[10] since $\bar{\sigma}_0 = \sigma_0$ and $\bar{\sigma}_1 = -\sigma_1, \ldots$

[11] In some textbooks these eqs. are considered as consequence of the *potential ansatz* for F: $F_{\mu\nu} = \frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu}$. But this ansatz would not be possible, if the homog. eqs. would not be fulfilled. So, in any case, two independent eqs. are needed to describe the em-field. The potential ansatz in *matrix form* reads $F = \frac{1}{2}(\bar{\partial}\mathbf{A} - \mathbf{\bar{A}}\partial)$.

[12] This matrix eq. actually consists of 4 complex, i.e. 8 real eqs.

<u>Proof</u>: Here F is a traceless matrix ($F + \bar{F} = 0$), which combines the field vectors, here as traceless, hermitean matrices $E = E^k\sigma_k, (k = 1, \ldots, 3)$ and $B = B^k\sigma_k$:

$$F = E + iB. \tag{11}$$

The *partial derivation operator* ∂ is according the above a hermitean (Minkowski-) matrix with the explicit form

$$\partial = \sigma_\mu \partial^\mu = \sigma_\mu \frac{\partial}{\partial x_\mu} = \frac{\partial}{\partial x_0} + \sigma_1 \frac{\partial}{\partial x_1} + \cdots = \frac{\partial}{\partial t} + \nabla. \tag{12}$$

Then the l.h.s. of eq. (10) can be decomposed into an hermitean and anti-hermitean term (vanishing, since $\mathbf{J}$ is hermitean), which are *both* Maxwell eqs.

$$\partial F = (\frac{\partial}{\partial t} + \nabla)(E + iB) = \underbrace{\dot{E} + \nabla \cdot E + i\nabla \times B}_{=\mathbf{J}} + \underbrace{\nabla \times E + i\dot{B} + i\nabla \cdot B}_{=0} \quad \underline{\text{q.e.d.}}$$

The Lorentz-*covariance* of (10) is guaranteed, when the following transformation rule for F is assumed[13]

$$F \rightarrow F' = \bar{T}^\dagger F T^\dagger. \tag{13}$$

For checking the mirror-invariance of (10) one must realize, that E, B transform as proper- and pseudo-vectors, resp. under spatial inversion: $E_{sp} = \bar{E} = -E$ and $B_{sp} = -\bar{B} = +B$. Thus $F_{sp} = -F^\dagger = \bar{F}^\dagger$ holds and consequently (10) is mirror-invariant.[14]

4.2 Lorentz-Force

In conventional component form the Lorentz-force is $K_\mu = F_{\mu\nu}J^\nu$. Here one has the matrix form, which obviously gives a hermitean force matrix $\mathbf{K}$:

$$\mathbf{K} = \frac{1}{2}(\mathbf{J}F + F^\dagger \mathbf{J}). \tag{14}$$

Of course, it is Lorentz-covariant and mirror-invariant.

4.3 Energy-Momentum-Tensor of Electromagnetic Field

Although it is not strictly necessary for the main thesis of this paper, I included this chapter, because it shows quite impressively the power of the matrix formalism.[15]

Inserting the Maxwell eq. (10) into the Lorentz-force (14) immediately gives:[16]

$$\underline{\mathbf{K}} = \frac{1}{2}((F^\dagger\partial)F + F^\dagger(\partial F)) = \frac{1}{2}\underline{F^\dagger \partial F} = \frac{\partial}{\partial x_\mu} \underbrace{\frac{1}{2}(F^\dagger \sigma_\mu F)}_{\stackrel{def}{=} \mathbf{T}_\mu} = \underline{\frac{\partial \mathbf{T}_\mu}{\partial x_\mu}}. \tag{15}$$

This derivation, consisting only of two simple reorderings, is significantly more concise than the corresponding component form (see e.g. [1], p. 50). Obviously the four hermitean matrices $\mathbf{T}_\mu$ (with 16 real components) here represent the **energy-momentum tensor**.

To get the corresponding component form, one uses the general mapping formula (5), which here leads to the 16 real components: $T_\mu^\nu = \frac{1}{2}\mathcal{T}(\mathbf{T}_\mu \sigma_\nu)$.[17]
Then with the following explicit formula the symmetry of $T_{\mu\nu} = T_{\nu\mu}$ can be easily shown, with usual formulas for the trace:

$$T_{\mu\nu} = \frac{1}{2}\mathcal{T}(\mathbf{T}_\mu \bar{\sigma}_\nu) = \frac{1}{4}\mathcal{T}(F^\dagger \sigma_\mu F \bar{\sigma}_\nu). \tag{16}$$

[13] Like necessary, for space rotations $\bar{T} = T^\dagger$ then E, B transform independently as 3-vectors, but for proper LT, they get mixed.

[14] From $\partial_{sp} F_{sp} = \mathbf{J}_{sp} \rightarrow \bar{\partial}\bar{F}^\dagger = \bar{\mathbf{J}}$, and after bar-operation and herm. conj. one gets the original eq. again, q.e.d.

[15] Of course, a general tensor with 16 real components, or a symmetric tensor with 10, cannot be represented by a single 2x2-matrix, but only by a set of matrices.

[16] the parentheses in the first terms denote the differential-operands of ∂, while in the underlined term it operates both to the left and right

[17] and $\mathbf{T}_\mu = T_\mu^\nu \sigma_\nu$

5 Relativistic Quantum Mechanics

In this section I will show, that relativistic quantum mechanics can be readily expressed with 2×2 matrices (which is well-known for a great part), but useless degrees of freedom are significantly supressed.

This is an especially important case, since in our contemporary understanding, quantum mechanics and esp. DIRACs eq. (with its various generalizations) is the fundament of the physical world. On the other hand, this theory is surely not yet finished, and it is to expect that new insights will evolve in the future, possibly within the framework of the matrix formalism.

Here closes the circle: the matrix formulation was first introduced for the description of quantum mechanical spin and can now hopefully lead to a better understanding of physics.

5.1 DIRAC-Equation

In most modern textbooks DIRACs eq. is presented in the conventional **component notation**, with the four CLIFFORD matrices γ_μ (and $\partial^\mu \stackrel{def}{=} \frac{\partial}{\partial x_\mu}$) for the 4-spinor wave function as column vector $\psi = (\psi_1, \ldots, \psi_4)^T$: (see [3], p. 50, [8] pp. 110)

$$i\gamma_\mu \partial^\mu \psi = m\psi. \tag{17}$$

This is a mathematically very elegant form, but it is achieved at the price of loss of physical reality of ψ. It exposes a great amount of ambiguity, since it is obviuously invariant under the so called *similarity transformations*[18] (see e.g. [3], p. 55):

$$\gamma_\mu \to U\gamma_\mu U^{-1} \quad \text{and} \quad \psi \to U\psi.$$

Here U is an arbitrary 4×4 matrix, containing 16 free complex parameters. This means, the formula (17) allows a linear transformation, leading to different representations, with 16 complex parameters without any change of the physical meaning. The components of ψ thus cannot represent any physical entities directly. In my opinion, this is a great disadvantage of this formula.

For the derivation of the **matrix form**, I start with the WEYL-*representation* of the γ_μ

$$\gamma_0 = \begin{pmatrix} 0, -I_2 \\ -I_2, 0 \end{pmatrix}, \quad \text{and} \quad \gamma_k = \begin{pmatrix} 0, & \sigma_k \\ -\sigma_k, 0 \end{pmatrix}, \quad k = 1, 2, 3. \tag{18}$$

This form has the important special feature, that here the 4-spinor can be decomposed into two 2-spinors Ψ, Φ: $\psi = \binom{\Psi}{\Phi}$, which transform independently under LORENTZ-transformations (see below), and (17) reads with them:

$$i\partial\Phi = -m\Psi \quad \text{and} \quad i\bar{\partial}\Psi = -m\Phi. \tag{19}$$

An additional, external electromagnetic vector potential field $\mathbf{A}$ is as usual introduced by the substitution $\partial \to \partial - ie\mathbf{A}$:

$$(i\partial + e\mathbf{A})\Phi = -m\Psi \quad \text{and} \quad (i\bar{\partial} + e\bar{\mathbf{A}})\Psi = -m\Phi. \tag{20}$$

This bi-spinor form of DIRACs eq. is well-known (although in most cases given in slightly different notation, see e.g. [8], p. 70) and sometimes referred to as "zigzag" model of the electron (e.g. [9]). From (20) the LORENTZ-transformation rules for the 2-spinors can be derived as:[19]

$$\Psi \to T\Psi \quad \text{and} \quad \Phi \to \bar{T}^\dagger \Phi, \tag{21}$$

leading to obviously covariant eqs. (20). Under spatial inversions both eqs. and consequently the spinors are interchanging: $\Psi \leftrightarrow \Phi$.

[18] They are not connected to a LORENTZ-transformation, since the spacetime components are not affected at all.

[19] consider again that ∂ and $\mathbf{A}$ transform like $\partial \to T\partial T^\dagger$

5.2 Matrix-Dirac-Equation

It is not yet commonly known, however, that both parts of (20) can be combined in one *single matrix equation.* This representation must be considered as the natural form of Diracs eq. in the MST context, and it opens up new possibilities for its generalization.

To develop this matrix eq., the second equation of (20) is converted in the following manner. With $M \overset{def}{=} i\partial + e\mathbf{A}$ it reads $\bar{M}\Psi = -m\Phi$.
Now, one uses the general formula for every 2x2 matrix M (M^T denoting transposed matrix):[20]

$$\bar{M} = \rho M^T \bar{\rho}, \quad \text{with} \quad \rho \overset{def}{=} \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix}$$

and inserting this leads to $M^T \rho\Psi = -m\rho\Phi$.
Of this one takes the complex conjugate, using $(M^T)^* = M^\dagger = -i\partial + e\mathbf{A}$:

$$(-i\partial + e\mathbf{A})\rho\Psi^* = -m\rho\Phi^*.$$

Here it is obviuously useful to define a new "tilde-operator"[21] for 2-spinors: $\underline{\widetilde{\Psi} \overset{def}{=} \rho\Psi^*}$ and the last eq. then writes $(-i\partial + e\mathbf{A})\widetilde{\Psi} = -m\widetilde{\Phi}$.
Then it is possible to combine this equation and the first of (20) as 2 columns into one 2x2 matrix equation:

$$e\mathbf{A}(\Phi, \widetilde{\Psi}) + i\partial(\Phi, -\widetilde{\Psi}) = -m(\Psi, \widetilde{\Phi}).$$

Now one defines the "spinor-matrix" $P \overset{def}{=} (\Phi, \widetilde{\Psi})$ (which is the replacement of the 4-spinor ψ) and notes $\bar{P}^\dagger = -(\Psi, \widetilde{\Phi})$, and with the constant matrix $S \overset{def}{=} \binom{i,\ 0}{0, -i}$ finally gets:

$$\underline{e\mathbf{A}P + \partial PS = m\bar{P}^\dagger}. \tag{22}$$

Although this formula at a first glance looks somewhat uncommon, esp. the right-side factor S in the derivation term, it possesses all features and solutions of the original 4-spinor equation (17).
The 2x2-matrix S together with the operator on the r.h.s here "magically absorb" all 4 Clifford matrices γ_μ. It should be clear from the above, that the special form of S ($S = i\sigma_3$), is the consequence of the choice of γ_μ. A more general form shall be discussed below.

To demonstrate the novel power of this matrix eq., one can derive an equivalent bilinear form by multiplicating it from left[22] with $P^\dagger$, resulting in

$$eP^\dagger\mathbf{A}P + P^\dagger(\partial P)S = m|P|^*. \tag{23}$$

Note, that this is still a matrix eq., although the r.h.s. is scalar ($\sim I$) and the l.h.s. terms are Lorentz-invariants. And it is still *equivalent* to (22), provided P is not singular ($|P| \neq 0$).
This direct way is only possible by using matrix algebra. By utilizing this bilinear form, esp. many computations, e.g. regarding gauge invariance, Lagrangian and conservations laws can be performed much simpler.

According to above definitions, P transforms consistently with $P \to \bar{T}^\dagger P$ under Lorentz-transformations and (22) is obviously covariant. Since $\bar{T}^\dagger$ operates only from the left on P, the two column 2-spinors of P transform equally and independently.
The *mirror-invariance* is guaranteed with $P \to P_{sp} = \bar{P}^\dagger$ (since $S = \bar{S}^\dagger$).

[20] A geometric explanation is, that the bar-operation as mentioned already, means spatial inversion, which is equal to the combined operation of transposing (i.e. $y \to -y$) and a rotation around y of 180°, which is performed by the transformation $T = \rho = i\sigma_2$.

[21] this operator obeys $\widetilde{\widetilde{\Psi}} = -\Psi$, since $\rho^2 = -1$

[22] multiplication from right produces another eq. with the same r.h.s.

Here also a *similarity transformation* is possible by right-side multiplication[23] of P with a matrix U obeying $U = \bar{U}^\dagger$

$$P \to PU \quad \text{and} \quad S \to U^{-1}SU, \tag{24}$$

but this 2x2-matrix U has only 2 free complex parameters (4 real), compared to 16 above (since one of the 4 real parameters is only a constant factor, there actually remain only 3 real free parameters).
Essentially this transformation says, that S (like U) can be any matrix obeying the condition $S = \bar{S}^\dagger$, which describes a subalgebra of matrices, which is isomorphic to the algebra of *quaternions*.
An obvious possibility to explain the remaining ambiguity physically, is discussed in chapter 5.4.

The *gauge invariance* of (22) and the corresponding Lagrangian (26) below is a bit different to the conventional form, because P *cannot* be multiplied with a scalar complex phase factor $e^{i\lambda}$, because the mass-term would then transform with $e^{-i\lambda}$. This impossibility to apply scalar phase factors is probably the reason, that this quite simple and obvious formula has never been considered before. Also the usual covariant replacement of the derivation operator $\partial^\mu \to D^\mu = \partial^\mu - ieA^\mu$ cannot simply be transcribed to $D = \partial - i\mathbf{A}$, but must be modified here.
However, one easily checks, that e.g. the gauge transformation

$$P \to Pe^{\lambda S} \quad \text{and} \quad e\mathbf{A} \to e\mathbf{A} + \partial\lambda, \tag{25}$$

where $\lambda(\mathbf{x})$ is an arbitrary real spacetime function, is the correct form.[24]
The matrix S can be thus seen as replacement of the imaginary unit i, since it also obeys $S^2 = -1$.

Stationary states, which represent bound states in atoms, are here similarly described by the ansatz $P = P_0(\mathbf{r})e^{-\varepsilon t S}$ (with ε as energy), which results in (since $\partial_t P = -\varepsilon PS$)

$$(\varepsilon + e\mathbf{A})P_0 + \nabla P_0 S = m\bar{P}_0^\dagger,$$

and it is easy to show, that it has the the same solutions as the original Dirac-eq.

An important special case regards massless, uncharged particles, esp. *neutrinos*. We know from experiments, that only *lefthanded* neutrinos exist, righthanded ones have never been observed. Diracs original eq., eg. written in the form (19), leaves this fact unexplained, because both parts decouple with $m = 0$, and so they have independent solutions for Ψ and Φ, representing both types of neutrinos.
The non-existence of righthanded neutrinos is a *direct consequence of the matrix eq.* (22), however. With $m = 0, e = 0$ it simplifies to $\partial PS = 0$ (for this singular case, (22) is no longer *equivalent* to the original Dirac-eq.). Here the factor S can be eliminated (by rhs multiplication with S^{-1}) giving $\underline{\partial P = 0}$. Their solutions can only be lefthanded particles, which is to show by transforming it into momentum space.

Another major advantage of the form (22) of Diracs eq should be shortly sketched at the end. Also *weak interactions* in the V-A-theory are most simply expressed in this form. This is seen as another hint, that this form is the primary one. It follows from the fact, that for the used Weyl-representation the matrix $\gamma_5 \overset{def}{=} i\gamma_0\gamma_1\gamma_2\gamma_3$ is a diagonal matrix: $\gamma_5 = \binom{I_2,\ 0}{0, -I_2}$. And since weak interaction couples in the 4-spinor form with $I_4 \pm \gamma_5$, so always in one of the eq-pair (20) the respective term vanishes.
Further considerations, regarding electro-weak gauge theory are done in chapter 5.4.

5.3 Lagrangian of Coupled Dirac- and EM-Field

Lagrangians play a very important role in modern field theory. They can readily be written in matrix form using the above entities. For the combined Dirac- and em-field it is the sum of four scalar terms:

$$\mathcal{L} = \mathcal{T}(P^\dagger(\partial P)S) + e\mathcal{T}(\mathbf{A}PP^\dagger) - 2\Re|F| - 2m\Re|P|. \tag{26}$$

This form demonstrates another advantage of the matrix representation. It can reveal subtle similarities between some terms (here e.g. the 3. and 4. term), which are hidden in the component form.

[23] left-side multiplication always describes a Lorentz-transformation
[24] Note that $e^{\lambda S}$ commutes with S and $\partial e^{\lambda S} = (\partial\lambda)e^{\lambda S}S$.

The validity of (26) can be proved by transforming it into component form, or better by deriving the field eqs., namely (10) and (22) from it. This complete derivation must be omitted here, only some basic steps should be stated.

In the first (differential) term, the partial operator should only operate to the right (as indicated by the parentheses). Furthermore one notices, that this term is not real (as normally required for a LAGRANGIAN and is the case for the other three terms). However, the actually relevant spacetime integral *is real*:

$$\mathcal{I} = \int d^4x\, \mathcal{T}(P^\dagger(\partial P)S) = \text{real}, \quad \text{i.e.} \quad \Im(I) = 0,$$

which is proved with the vanishing of the integral $\int \mathcal{T}((P^\dagger \partial P)S) = \int \partial_\mu \mathcal{T}(P^\dagger \sigma_\mu PS) = 0$ by GAUSS*' law* and partial integration.

The third term is the well-known LAGRANGIAN of the electromagnetic field $\mathcal{L}_{em} = -2\Re|F| = E^2 - B^2$, since $F = \frac{1}{2}(\bar{\partial}\mathbf{A} - \bar{\mathbf{A}}\partial) = E + iB$.

Consequently, the variation of $\mathbf{A}$ in the second and third term, leads to MAXWELLs eq. (10), if the 4-current of the DIRAC-field is defined as

$$\mathbf{J}_e \stackrel{def}{=} e\overline{PP^\dagger} \quad (= e\bar{P}^\dagger\bar{P}). \tag{27}$$

Variation of $P^\dagger$ (or independently P) in the terms 1, 2 and 4 leads to DIRACs-eq. (22).

5.4 YANG-MILLS Gauge-Fields in Matrix Form

The remaining possibility of similarity transformations $P \to PU$ ($\bar{U} = U^\dagger$, see chapter 5.1) leads to an obvious generalization of the DIRAC eq. in matrix form (22) with four vector fields $\mathbf{B}^\mu$, $\mu = 0, \ldots, 3$:

$$\partial PS + \mathbf{B}^\mu P\sigma_\mu = m\bar{P}^\dagger. \tag{28}$$

This also resolves the remaining ambiguities of P. Here $\mathbf{B}^0 \equiv \mathbf{A}$ is apparently again the em. vector potential, which is invariant under this transformation. The other three fields $(\mathbf{B}^1, \mathbf{B}^2, \mathbf{B}^3)$ mix however (under LORENTZ-trafos they act as normal MMs, like $\mathbf{A}$), since with $U\sigma_k U^{-1} = a_k^m \sigma_m$, $(m, k = 1, 2, 3)$ follows $\mathbf{B}^k \to a_m^k \mathbf{B}^m$. With the additional restriction $|U| = 1$ this is the $SU(2)$ group and its $SO(3)$ representation acts on the $\mathbf{B}^k$.

One can now formulate the fascinating hypothesis, that by introducing a local non-abelian gauge field $U(\mathbf{x})$ the unified electro-weak field may be represented, similar to YANG-MILLS theory (see e.g. [2]).

It is striking, that this gauge field shows remarkable similarities to the symmetry $SU(2) \times U(1)$ as proposed by WEINBERG and SALAM for the unified theory, although it is evidently not equivalent.[25]
To get all 4 gauge fields, it is obviously necessary to use the infinitesimal generator of the complete quaternionic algebra[26] for U instead of the subset $SU(2)$, which is:

$$U(\mathbf{x}) = e^{\lambda^0\sigma_0 + i\lambda^k\sigma_k} \approx I + \lambda^0\sigma_0 + i\lambda^k\sigma_k,\ k = 1, 2, 3,$$

with 4 real spacetime functions $\lambda^\mu(\mathbf{x})$, $(|\lambda^\mu| \ll 1)$.
Then $\lambda^3(\mathbf{x})$ represents the em. gauge field, coupled with $\mathbf{A} \equiv \mathbf{B}^0$ (if $S = i\sigma_3$, as before (25) explained). The other three gauge fields, however, lead to anti-hermitean vector fields $(\mathbf{B}^k)^\dagger = -\mathbf{B}^k$, because $i\partial\lambda^k$ is anti-hermitean.
The gauge fields $\lambda^{1,2}(\mathbf{x})$ couple with $\pm\mathbf{B}^{2,1}$, respectively. The gauge field $\lambda^0(\mathbf{x})$ obviously represents a boost (since $|U| \neq 1$) and couples with $\mathbf{B}^3$, very different to the standard theory.

In conclusion should be emphasized the remarkable fact, that the gauge symmetry here is an *intrinsic* feature of DIRACs eq. in matrix form and its group structure is automatically determined. Moreover, gauge- and LORENTZ-symmetry here turn out to be "two sides of one coin" in the general transformation formula for the spinor-matrix $P \to TPU$.

[25] Here $U(1)$ for electromagnetic gauge is a subgroup of $SU(2)$.

[26] See again chapter 5.1; the only required condition for U is actually $\bar{U} = U^\dagger$, which is fulfilled by this algebra. Remember, that $\sigma_0 = I$, so one gets $\bar{U} = U^\dagger = e^{\lambda^0 - i\lambda^k\sigma_k}$

Further discussions, regarding covariant field equations for the associated generalized em. field tensors F^μ, the complete LAGRANGIAN and a possible HIGGS mechanism for symmetry breaking, go beyond the scope of this article and shall be considered in a subsequent paper.

5.5 2-Spinors and MINKOWSKI-Matrices

At the end, some general remarks about the relations of spinors and matrices should be added. As stated above, 2-spinors are represented by binary column matrices $\Psi = \binom{\alpha}{\beta}, \Phi = \binom{\gamma}{\delta}, \ldots$, which transform under LT as $\Psi \to T\Psi$. Then for a any pair of spinors $P = (\Psi, \Phi)$ the determinant $|P| = |\Psi, \Phi| = \alpha\delta - \beta\gamma$ is obviously a LORENTZ-invariant, because $TP = (T\Psi, T\Phi)$.

Also note the important fact, that spinor products, like e.g. the matrix:[27]

$$\mathbf{H} = \Psi\Psi^\dagger = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}(\alpha^*, \beta^*) = \begin{pmatrix}|\alpha|^2, \alpha\beta^* \\ \beta\alpha^*, |\beta|^2\end{pmatrix} \tag{29}$$

is obviously a MINKOWSKI-matrix (in this special example a null-matrix: $|\mathbf{H}| = 0$). That says, that matrices can be constructed by spinors, but the opposite does not hold. Only null-matrices can be uniquely (up to a phase-factor) decomposed into spinors.

It is also a fascinating feature of forms like (29), that they have a positive definite time-component, which might help to explain the direction of time. From the realization, that 2-spinors are the algebraic basis of all, it will be possibly feasible to develop a complete theory of spacetime only with spinors.

The crucial problem is however, how to retain the spacetime translation symmetry in such constructs.

6 Conclusions

In this paper I have presented the most important concepts of Special Relativity in 2x2-matrix form, namely the entities and equations of electromagnetic interactions and the DIRAC equation. Essentially this form uses another algebraic concept of spacetime, rather than the conventional vector space.

Although the equations are obviously equivalent to conventional component formulation, I have showed that the matrix form has several striking advantages, which suggest that this form should be considered as the primary description of the physical world.

The main advantages can be shortly summarized:

- The metric tensor needs not to be postulated and spacetime can have no more than four dimensions
- The MAXWELL equations are represented by a single equation rather than two independent
- The DIRAC spinor field in the novel DIRAC eq. in matrix form has much less of degrees of freedom without any physical meaning and this form explains the non-existence of righthanded neutrinos
- A new type of YANG-MILLS gauge fields arises from the generalization of this matrix DIRACs eq., which possibly can describe electro-weak interactions

From a heuristic point of view, from a bunch of theories which describe the same phenomena with equal accuracy, the one with the least prerequisites should be given preference.

Another major intention of writing this paper was, to encourage other theoretical physicists, to find extensions of this concept for new theories. Also I hope to be able, to present a new concept for quantum mechanics on the basis of this algebra, which can replace the wave function by a discrete model. Some first ideas can be found in [6].

[27] In most textbooks a "dotted index" notation is used to describe conjugated spinors like $\Psi^\dagger$, that goes back to the first publications on this topic. I do not adopt it here.

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